SOME REMARKS ON SINC INTEGRALS AND THEIR CONNECTION WITH COMBINATORICS, GEOMETRY AND PROBABILITY

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ABSTRACT. We give an alternative, combinatorial/geometrical evaluation of a class of improper sinc integrals studied by the Borweins. A probabilistic interpretation is also noted and used to shed light on a related combinatorial identity.

1. INTRODUCTION

In [1], among other things the Borweins study integrals of the form

$$I_n := \int_0^\infty \prod_{j=0}^n \operatorname{sinc}(a_j x)\, dx, \tag{1.1}$$

where $\operatorname{sinc}(x) := (\sin x)/x$ for $x \neq 0$ and $\operatorname{sinc}(0) := 1$. They use a version of the Parseval/Plancherel formula to prove that, subject to certain conditions on the parameters a_j, the sequence $I_1, I_2, \ldots$ is decreasing; they also give an explicit evaluation of I_n by expanding the product of sines into a sum of cosines and then integrating by parts. As the Borweins observed, the integral (1.1) can be interpreted geometrically as the volume of an n-dimensional polyhedron obtained by cutting an n-dimensional hypercube by two $(n-1)$-dimensional hyperplanes, and it is interesting to note that their formula is essentially a signed sum over the vertices of the hypercube. It may therefore be of interest to give an independent evaluation of (1.1) using methods from combinatorial geometry. The derivation is provided in §3.

In [2], related integrals arise in a probabilistic setting—namely the problem of determining the probability density of a sum of independent random variables uniformly distributed in different ranges. We employ this interpretation here to give an alternative proof of an elegant combinatorial identity related to the evaluation of I_n using methods from probability theory. See Proposition 1 below.

2. Fourier Background and Connection with Probability Theory

For our purposes, it is more convenient to take products from 1 to n, as opposed to 0 to n in (1.1). Thus, we fix a positive integer n, and let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a vector of positive real numbers. For each $j = 1, 2, \ldots, n$, define a step function $\chi_j : \mathbf{R} \to \mathbf{R}$ by

$$2a_j\,\chi_j(x) = \begin{cases} 1 & \text{if } |x| < a_j, \\ \frac{1}{2} & \text{if } |x| = a_j, \\ 0 & \text{if } |x| > a_j. \end{cases} \tag{2.1}$$

Let $f_n : \mathbf{R} \to \mathbf{R}$ denote the n-fold convolution $\chi_1 * \chi_2 * \cdots * \chi_n$, so that for all real x,

$$f_n(x) = \int_{-\infty}^{\infty} \chi_1(x-y_2) \int_{-\infty}^{\infty} \chi_2(y_2-y_3) \cdots \int_{-\infty}^{\infty} \chi_{n-1}(y_{n-1}-y_n)\chi_n(y_n)\,dy_2 \ldots dy_n.$$

It is now easy to see that $f_n(x)$ is simply the volume (in the sense of Lebesgue measure on $\mathbf{R}^{n-1}$) of the $(n-1)$-dimensional polyhedron

$$\{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : \textstyle\sum_{j=1}^n x_j = x \text{ and } |x_j| < a_j \text{ for } j = 1, 2, \ldots, n\} \tag{2.2}$$

divided by the volume $\prod_{j=1}^n (2a_j)$ of the n-dimensional hypercube

$$\{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : |x_j| < a_j \quad \text{for} \quad j = 1, 2, \ldots, n\}. \tag{2.3}$$

The volume of the region (2.2) will be evaluated directly in §3; but first we connect this problem with that of evaluating integrals such as (1.1). The Fourier transform of f_n is given by

$$\widehat{f_n}(t) = \prod_{j=1}^n \widehat{\chi_j}(t) = \prod_{j=1}^n \int_{-\infty}^{\infty} e^{itx}\chi_j(x)\,dx = \prod_{j=1}^n \operatorname{sinc}(a_j t), \qquad t \in \mathbf{R}. \tag{2.4}$$

Since for all real x, $f_n(x) = \frac{1}{2}f_n(x+) + \frac{1}{2}f_n(x-)$ by continuity of f_n for $n > 1$, and by definition of χ_1 when $n = 1$, Fourier inversion gives

$$f_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\widehat{f_n}(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \prod_{j=1}^n \operatorname{sinc}(a_j t)\,dt, \qquad x \in \mathbf{R}, \tag{2.5}$$

where we have made the change of variable $t \mapsto -t$ and used the fact that sinc is even. We remark that apart from our convention of starting products at $j = 1$, $f_n(0)$ coincides with the Borweins' integral (1.1) up to a numerical factor. More generally, $f_n(x)$ falls under the scope of their Theorem 2(ii). However, to facilitate the presentation of our combinatorial evaluation, it is convenient to reformulate their result as follows. We first make the following

Definition 1. Let $\tau : \mathbf{R} \to \mathbf{R}$ be given by

$$\tau(x) = \begin{cases} 1, & \text{if } x > 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{2.6}$$

and for x real and n a positive integer, let $x_+^{n-1} := x^{n-1}\tau(x)$. Note that $x_+^0 = \tau(x)$ and $x_+^n = (\max(x,0))^n$ for $n > 0$.

We are now ready to state our reformulation.

Theorem 1. *Let n be a positive integer, let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a vector of positive real numbers, and let $f_n(x)$ be as in (2.5). Then for all real x,*

$$f_n(x) = \left[\sum_{\vec{\varepsilon} \in \{-1,1\}^n} (x + \vec{\varepsilon} \cdot \vec{a})_+^{n-1} \prod_{j=1}^{n} \varepsilon_j \right] \Bigg/ \left[(n-1)! \prod_{j=1}^{n} (2a_j) \right], \qquad (2.7)$$

in which the sum is over all 2^n vectors of signs $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ with $\varepsilon_j = \pm 1$ for each $j = 1, 2, \ldots, n$, and of course $\vec{\varepsilon} \cdot \vec{a}$ denotes the inner product $\sum_{j=1}^{n} \varepsilon_j a_j$.

To see more directly the connection with Theorem 2(ii) of [1], simply replace the τ function in Theorem 1 by the sign function using the relationship $\text{sign}(x) = 2\tau(x) - 1$ for x real. It then suffices to prove the identity

$$\sum_{\vec{\varepsilon} \in \{-1,1\}^n} (x + \vec{\varepsilon} \cdot \vec{a})^{n-1} \prod_{j=1}^{n} \varepsilon_j = 0, \qquad x \in \mathbf{R}, \qquad (2.8)$$

which is a special case of [1, Theorem 2(i)], there proved most elegantly using the method of generating functions. We conclude this section with an alternative proof of (2.8) based on probabilistic considerations. For the reader's convenience, the result is restated as

Proposition 1 (Theorem 2(i) of [1]). *Let n be a positive integer and $\vec{a} = (a_1, a_2, \ldots, a_n)$ a vector of positive real numbers. Then for all real x, we have*

$$\sum_{\vec{\varepsilon} \in \{-1,1\}^n} (x + \vec{\varepsilon} \cdot \vec{a})^r \prod_{j=1}^{n} \varepsilon_j = \begin{cases} 0, & \text{if } r = 0, 1, 2, \ldots, n-1, \\ n! \, 2^n \prod_{j=1}^{n} a_j, & \text{if } r = n. \end{cases}$$

Proof. As in [2], we note that $f_n(x)$ represents the probability density of the sum of n independent random variables $X_1, X_2, \ldots, X_n$ with X_j uniformly distributed in the interval $[-a_j, a_j]$ for each $j = 1, 2, \ldots, n$. In light of Theorem 1, the cumulative distribution $F_n(x) := \Pr(\sum_{j=1}^{n} X_j \le x)$ is given by

$$F_n(x) = \int_{-\infty}^{x} f_n(y) \, dy = \left[\sum_{\vec{\varepsilon} \in \{-1,1\}^n} (x + \vec{\varepsilon} \cdot \vec{a})_+^{n} \prod_{j=1}^{n} \varepsilon_j \right] \Bigg/ \left[n! \prod_{j=1}^{n} (2a_j) \right]. \quad (2.9)$$

Let $A_n := \sum_{j=1}^{n} a_j$. Since each X_j is uniformly distributed in $[-a_j, a_j]$, the sum $\sum_{j=1}^{n} X_j$ must fall within the interval $[-A_n, A_n]$, whence $F_n(x) = F_n(\infty) = \Pr(\sum_{j=1}^{n} X_j \le A_n) = 1$

for all $x \geq A_n$. But if $x \geq A_n$, then $x + \vec{\varepsilon} \cdot \vec{a} \geq 0$ for each $\vec{\varepsilon} \in \{-1, 1\}^n$, and so for such x we can drop the subscripted "+" from (2.9). It follows that

$$\sum_{\vec{\varepsilon} \in \{-1,1\}^n} (x + \vec{\varepsilon} \cdot \vec{a})^n \prod_{j=1}^{n} \varepsilon_j = n! \prod_{j=1}^{n} (2a_j) = n! \, 2^n \prod_{j=1}^{n} a_j \qquad (2.10)$$

holds for all $x \geq A_n$. Since the left hand side of (2.10) is a polynomial of degree at most n in x which is constant for all sufficiently large values of x, it must in fact be constant for all real x by the identity theorem. This proves the case $r = n$ of the proposition. The other cases follow by repeatedly differentiating with respect to x. $\qquad \square$

Remark 1. A related probabilistic interpretation of the integral (1.1), which is essentially our $f_n(0)$, can be found in [1, p. 81, Remarks 1(b)(ii)].

3. Combinatorial Proof of Theorem 1

We will prove the formula (2.7) of Theorem 1 by evaluating the volume of the polyhedron (2.2) directly. Let us denote this volume by $V_n(x)$. Since the case $n = 1$ is trivial, we will assume $n > 1$. Let the vector $\vec{a}$ be as in the previous section, and let N be the set of vectors in $\mathbf{R}^n$ whose respective components exceed the components of $-\vec{a}$. Thus, $N := \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_j > -a_j \text{ for } j = 1, 2, \ldots, n\}$. To each subset A of N, we associate a non-negative function $w(A) : \mathbf{R} \to \mathbf{R}$ defined by

$$w(A)(x) := \text{Vol}\{(x_1, x_2, \ldots, x_n) \in A : \sum_{j=1}^{n} x_j = x\},$$

where, as in (2.2), the volume is taken in the sense of Lebesgue measure on $\mathbf{R}^{n-1}$. Clearly w is a non-negative additive weight-function on the subsets of N; for if A and B are disjoint subsets of N, then $w(A \cup B) = w(A) + w(B)$, and $w(A) \geq 0$ for all $A \subseteq N$. Therefore, if we let $S := \{1, 2, \ldots, n\}$ and $R_k := \{(x_1, x_2, \ldots, x_n) \in N : x_k > a_k\}$ for $k \in S$, then

$$\bigcap_{j \in S} (N \setminus R_j) = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : -a_j < x_j \leq a_j \text{ for } j = 1, 2, \ldots, n\},$$

and hence in view of (2.2) and the definition of w, we have for every $x \in \mathbf{R}$,

$$V_n(x) = w\left[\bigcap_{j \in S} (N \setminus R_j)\right](x), \qquad (3.1)$$

since the underlying sets differ by a set of measure zero in $\mathbf{R}^{n-1}$. The Inclusion-Exclusion Principle states that

$$w\left[\bigcap_{j \in S} (N \setminus R_j)\right] = \sum_{T \subseteq S} (-1)^{|T|} w\left(\bigcap_{j \in T} R_j\right), \qquad (3.2)$$

where the sum is over all subsets T of S, $|T|$ denotes the cardinality of T, and under the usual convention for intersections over empty sets, the term corresponding to $T = \{\}$ is

$w(N)$. Fortunately, the weights of the sets R_k and their intersections are relatively easy to compute. We make a linear change of variable

$$y_j := \sum_{k=j}^{n} x_k, \qquad j = 1, 2, \ldots, n.$$

Then $y_j = y_{j+1} + x_j$ for $j = 1, 2, \ldots, n-1$, $y_n = x_n$, and the Jacobian of the transformation is 1. Given $T \subseteq S$, let $\vec{b} = \vec{b}(T) = (b_1, b_2, \ldots, b_n)$ be the vector whose jth component b_j is equal to a_j if $j \in T$, and $b_j = -a_j$ otherwise. Then

$$R(T) := \bigcap_{j \in T} R_j = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_j > b_j \quad \text{for} \quad j = 1, 2, \ldots, n\}.$$

For each $j = 1, 2, \ldots, n$, let $c_j := \sum_{k=j}^{n} b_j$. If $\vec{x} \in R(T)$, then $y_j > c_j$, and $y_j - b_j > y_{j+1} > c_{j+1}$ for $j = 1, 2, \ldots, n-1$. Thus,

$$w(R(T))(y_1) = \int_{c_2}^{y_1 - b_1} \int_{c_3}^{y_2 - b_2} \cdots \int_{c_n}^{y_{n-1} - b_{n-1}} dy_n \cdots dy_3 \, dy_2.$$

Now make the change of variable $z_j = y_j - c_j$ for $j = 1, 2, \ldots, n$. Then each $z_j > 0$, and we obtain

$$w(R(T))(y_1) = \int_0^{z_1} \int_0^{z_2} \cdots \int_0^{z_{n-1}} dz_n \cdots dz_3 \, dz_2 = \frac{z_1^{n-1}}{(n-1)!}.$$

In light of the constraint $y_1 - c_1 = z_1 > 0$, we see that

$$w\Big(\bigcap_{j \in T} R_j\Big)(y_1) = w(R(T))(y_1) = \frac{(y_1 - c_1)_+^{n-1}}{(n-1)!} = \frac{(y_1 - \sum_{j=1}^{n} b_j)_+^{n-1}}{(n-1)!}.$$

Thus, from (3.1) and (3.2) it follows that

$$V_n(x) = \sum_{T \subseteq S} (-1)^{|T|} w\Big(\bigcap_{j \in T} R_j\Big)(x) = \frac{1}{(n-1)!} \sum_{\vec{\varepsilon} \in \{-1,1\}^n} (x + \vec{\varepsilon} \cdot \vec{a})_+^{n-1} \prod_{j=1}^{n} \varepsilon_j,$$

where we have made the correspondence $\varepsilon_j = -1$ if $j \in T$ and $\varepsilon_j = 1$ if $j \notin T$ in each term of the sums. Since $f_n(x) = V_n(x) / \prod_{j=1}^{n}(2a_j)$, the proof of Theorem 1 is complete. $\square$

References

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